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EXISTENCE RESULTS FOR A QUASI-LINEAR DIFFERENTIAL PROBLEM

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The aim of this paper is to establish the existence of at least one non-trivial solution for Neumann quasi-linear problems. Our approach is based on variational methods.

1. Introduction

The aim of this paper is to ensure the existence of at least one non-trivial solution for the following Neumann boundary value problem

$$\begin{cases} -u'' + uh(u') = \lambda\alpha(x)f(u)h(u') \\ u'(a) = u'(b) = 0, \end{cases} \tag{N_λ}$$

where $\alpha : [a,b] \to \mathbb{R}$ is a positive continuous function, $f : \mathbb{R} \to \mathbb{R}$ and $h : \mathbb{R} \to \mathbb{R}$ are continuous functions and λ is a positive real parameter.

Existence and multiplicity of solutions for Neumann boundary value problems have been investigated by several authors and, for an overview on this subject, we refer to [1], [3] - [6], [8], [9], [11] - [14].

The main result of this paper is Theorem 3.1, which generalizes [6, Theorem 3.1] to the case where the nonlinear term is not constant with respect to u'. Two relevant consequences of Theorem 3.1 (that is, Corollary 3.2 and Theorem

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3.3) are also pointed out. Here, as an example, we presented a special case of our main result.

Theorem 1.1. *Let* $\alpha : [a,b] \to \mathbb{R}$ *be a nonnegative continuous function and* $f,h : \mathbb{R} \to \mathbb{R}$ *be continuous functions. Suppose that* h *is bounded and strictly positive, and that*

$$\lim_{\xi \to 0^+} \frac{f(\xi)}{\xi} = +\infty.$$

Then, there exists λ^* *such that, for each* $\lambda \in]0,\lambda^*[$, *the problem* (N_λ) *admits at least one positive classical solution.*

Our approach is based on a critical point theorem obtained in [2] (see Theorem 2.1).

The paper is arranged as follows: in Section 2, we recall some basic definitions and our main tool, while Section 3 is devoted to our main results.

2. Preliminaries and basic notations

Our main tool is the Ricceri variational principle [10, Theorem 2.5] as given in [2, Theorem 5.1] which is below recalled (see also [2, Proposition 2.1] and [7, Theorem 2.1]). First, given $\Phi, \Psi : X \to \mathbb{R}$, put

$$\beta(r_1,r_2) = \inf_{v \in \Phi^{-1}(]r_1,r_2[)} \frac{\sup\limits_{u \in \Phi^{-1}(]r_1,r_2[)} \Psi(u) - \Psi(v)}{r_2 - \Phi(v)} \tag{1}$$

and

$$\rho_2(r_1,r_2) = \sup_{v \in \Phi^{-1}(]r_1,r_2[)} \frac{\Psi(v) - \sup\limits_{u \in \Phi^{-1}(]-\infty,r_1])} \Psi(u)}{\Phi(v) - r_1}, \tag{2}$$

for all $r_1, r_2 \in \mathbb{R}$, with $r_1 < r_2$.

Theorem 2.1. *([2, Theorem 5.1]) Let* X *be a reflexive real Banach space;* $\Phi : X \to \mathbb{R}$ *be a sequentially weakly lower semicontinuous, coercive and continuously Gâteaux differentiable function whose Gâteaux derivative admits a continuous inverse on* X^*; $\Psi : X \to \mathbb{R}$ *be a continuously Gâteaux differentiable function whose Gâteaux derivative is compact. Put* $I_\lambda = \Phi - \lambda\Psi$ *and assume that there are* $r_1, r_2 \in \mathbb{R}$, *with* $r_1 < r_2$, *such that*

$$\beta(r_1,r_2) < \rho_2(r_1,r_2), \tag{3}$$

where β *and* ρ_2 *are given by (1) and (2).*
Then, for each $\lambda \in \left] \frac{1}{\rho_2(r_1,r_2)}, \frac{1}{\beta(r_1,r_2)} \right[$ *there is* $u_{0,\lambda} \in \Phi^{-1}(]r_1,r_2[)$ *such that* $I_\lambda(u_{0,\lambda}) \le I_\lambda(u)$ *for all* $u \in \Phi^{-1}(]r_1,r_2[)$ *and* $I'_\lambda(u_{0,\lambda}) = 0$.

Let X be the Sobolev space $W^{1,2}([a,b])$ endowed with the norm

$$\|u\| := \left(\int_a^b |u'(x)|^2 dx + \int_a^b |u(x)|^2 dx\right)^{\frac{1}{2}}.$$

Throughout the sequel, $f : \mathbb{R} \to \mathbb{R}$ is a continuous function, $h : \mathbb{R} \to \mathbb{R}$ is a positive continuous function, $\alpha : [a,b] \to \mathbb{R}$ is a sommable function and λ is a positive real parameter. Put

$$F(t) = \int_0^t f(\xi)d\xi, \quad \text{for all } t \in \mathbb{R},$$

$$F_1(x,t) = \int_0^t \alpha(x)f(\xi)d\xi = \alpha(x)F(t), \qquad \text{for all } (x,t) \in [a,b] \times \mathbb{R},$$

and, put

$$H(y) = \int_0^y (\int_0^\sigma \frac{1}{h(\tau)} d\tau) d\sigma, \qquad \text{for all } y \in \mathbb{R}.$$

We recall that $u : [a,b] \to \mathbb{R}$ is called weak solution of Problem (N_λ) if $u \in W^{1,2}([a,b])$ and

$$\int_a^b H'(u'(x))v'(x)dx + \int_a^b u(x)v(x)dx = \lambda \int_a^b \alpha(x)f(u(x))v(x)dx,$$

for all $v \in W^{1,2}([a,b])$.
We also recall that a weak solution is a generalized solution, that is,
$u \in C^1([a,b])$, $u' \in AC([a,b])$, $-u''(x) + u(x)h(u'(x)) = \lambda\alpha(x)f(u(x))h(u'(x))$, for a.e. $x \in [a,b]$, and $u'(a) = u'(b) = 0$.
Moreover, if α is continuous, each weak solution is a classical solution, that is, $u \in C^2([a,b])$, $-u''(x) + u(x)h(u'(x)) = \lambda\alpha(x)f(u(x))h(u'(x))$ for all $x \in [a,b]$, and $u'(a) = u'(b) = 0$. Finally, put

$$\gamma = \left(\max\left\{2(b-a); \frac{2}{b-a}\right\}\right)^{\frac{1}{2}},$$

we recall the following inequality which we use in the sequel

$$\max_{x\in[a,b]} |u(x)| \leq \gamma\|u\|, \tag{4}$$

for all $u \in X$ and for all $x \in [a,b]$.

3. Main Results

In this Section, we establish existence results for the Neumann boundary value problem (N_λ).

Given two positive constants m, M, with $m \leq M$, put

$$\delta_1 = \left(\min\left\{\frac{1}{M(b-a)};\frac{1}{b-a}\right\}\right)^{\frac{1}{2}},\ \delta_2 = \left(\max\left\{\frac{1}{m(b-a)};\frac{1}{b-a}\right\}\right)^{\frac{1}{2}}.$$

Moreover, given three nonnegative constants c_1, c_2, d, with $\delta_1 c_1 < \gamma d < \delta_2 c_2$, put

$$a(c_2,d) := \frac{\max_{|t|\leq c_2} F(t) - F(d)}{\delta_2^2 c_2^2 - \gamma^2 d^2}$$

and

$$b(c_1,d) := \frac{F(d) - \max_{|t|\leq c_1} F(t)}{\gamma^2 d^2 - \delta_1^2 c_1^2}.$$

We give our main result.

Theorem 3.1. *Let $\alpha : [a,b] \to \mathbb{R}$ be a nonnegative function and let $f,h : \mathbb{R} \to \mathbb{R}$ be continuous functions. Assume that there exist two positive constants m, M, such that*

(i) $m \leq h(y) \leq M$, *for all* $y \in \mathbb{R}$,

and, assume that there exist three nonnegative constants c_1, c_2, d, with $\delta_1 c_1 < \gamma d < \delta_2 c_2$, such that

$$a(c_2,d) < b(c_1,d). \tag{5}$$

Then, for each $\lambda \in \left]\frac{b-a}{2\gamma^2\|\alpha\|_1 b(c_1,d)}, \frac{b-a}{2\gamma^2\|\alpha\|_1 a(c_2,d)}\right[$, *the problem (N_λ) admits at least one weak solution $\overline{u}$, such that* $\frac{c_1}{\gamma} < \|\overline{u}\| < \frac{c_2}{\gamma}$.

Proof. Put

$$\Phi(u) := \frac{1}{2}\int_a^b |u(x)|^2 dx + \int_a^b H(u'(x))dx,$$

$$\Psi(u) := \int_a^b F_1(x,u(x))dx,$$

for all $u \in X$.

It is well known that Φ and Ψ satisfy all regularity assumptions requested in

Theorem 2.1 and that the critical points in X of the functional $\Phi - \lambda\Psi$ are exactly the weak solutions of the problem (N_λ). By using *(i)*, one has

$$\min\left\{\frac{1}{2M};\frac{1}{2}\right\}\|u\|^2 \le \Phi(u) \le \max\left\{\frac{1}{2m};\frac{1}{2}\right\}\|u\|^2,$$

for every $u \in X$. Our aim is to apply Theorem 2.1. To this end, put

$$r_1 = \frac{b-a}{2}\frac{\delta_1^2}{\gamma^2}c_1^2, \quad r_2 = \frac{b-a}{2}\frac{\delta_2^2}{\gamma^2}c_2^2$$

and

$$u_0(x) = d, \quad \text{for all } x \in [a,b].$$

Clearly, $u_0 \in X$ and one has

$$\Phi(u_0) = \frac{1}{2}\int_a^b |u_0|^2 dx + \int_a^b H(u_0')dx = \frac{1}{2}d^2(b-a),$$

$$\Psi(u_0) = \int_a^b F_1(x,u_0(x))dx = \|\alpha\|_1 F(d),$$

where

$$\|\alpha\|_1 := \int_a^b |\alpha(x)|dx.$$

From $\delta_1 c_1 < \gamma d < \delta_2 c_2$, one has $r_1 < \Phi(u_0) < r_2$. Moreover, for all $u \in X$ such that $\Phi(u) < r_2$, taking (4) into account, one has

$$|u(x)| < c_2, \quad \text{for all } x \in [a,b],$$

and

$$\int_a^b F_1(x,u(x))dx \le \int_a^b \max_{|t|\le c_2} F_1(x,t)dx = \|\alpha\|_1 \max_{|t|\le c_2} F(t).$$

Therefore

$$\sup_{u\in\Phi^{-1}(]-\infty,r_2[)} \Psi(u) \le \|\alpha\|_1 \max_{|t|\le c_2} F(t).$$

Arguing as before, we obtain

$$\sup_{u\in\Phi^{-1}(]-\infty,r_1])} \Psi(u) \le \|\alpha\|_1 \max_{|t|\le c_1} F(t).$$

Therefore, one has

$$\begin{aligned}\beta(r_1,r_2) &\le \frac{\sup\limits_{u\in\Phi^{-1}(]-\infty,r_2[)} \Psi(u) - \Psi(u_0)}{r_2 - \Phi(u_0)} \le \\ &\le \frac{2\gamma^2\|\alpha\|_1}{b-a}\frac{\max\limits_{|t|\le c_2} F(t) - F(d)}{\delta_2^2 c_2^2 - \gamma^2 d^2} = \frac{2\gamma^2\|\alpha\|_1}{b-a}a(c_2,d). \qquad (6)\end{aligned}$$

On the other hand, one has

$$\rho_2(r_1,r_2) \geq \frac{\Psi(u_0) - \sup\limits_{u\in\Phi^{-1}(]-\infty,r_1])}\Psi(u)}{\Phi(u_0)-r_1} \geq$$

$$\geq \frac{2\gamma^2\|\alpha\|_1}{b-a}\frac{F(d)-\max\limits_{|t|\leq c_1}F(t)}{\gamma^2d^2-\delta_1^2c_1^2} = \frac{2\gamma^2\|\alpha\|_1}{b-a}b(c_1,d). \tag{7}$$

Hence, from (5) one has

$$\beta(r_1,r_2) < \rho_2(r_1,r_2).$$

Therefore, owing to Theorem 2.1, for each

$$\lambda \in \left]\frac{b-a}{2\gamma^2\|\alpha\|_1 b(c_1,d)}, \frac{b-a}{2\gamma^2\|\alpha\|_1 a(c_2,d)}\right[,$$

$\Phi - \lambda\Psi$ admits at least one critical point $\bar{u}$ such that

$$r_1 < \Phi(\bar{u}) < r_2,$$

that is

$$\frac{c_1}{\gamma} < \|\bar{u}\| < \frac{c_2}{\gamma}.$$

Hence, the proof is complete. □

Now, we point out the following consequence of Theorem 3.1.

Corollary 3.2. *Let $\alpha : [a,b] \to \mathbb{R}$ be a nonnegative function, $h : \mathbb{R} \to]0,+\infty[$ and $f : \mathbb{R} \to [0,+\infty[$ be continuous functions. Assume that (i) holds and that there exist two positive constants c,d, with $c > \frac{\gamma}{\delta_2}d$, such that*

$$\frac{F(c)}{c^2} < \left(\frac{\delta_2}{\gamma}\right)^2 \frac{F(d)}{d^2}. \tag{8}$$

Then, for each $\lambda \in \left]\frac{b-a}{2\|\alpha\|_1}\frac{d^2}{F(d)}, \frac{b-a}{2\|\alpha\|_1}\left(\frac{\delta_2}{\gamma}\right)^2\frac{c^2}{F(c)}\right[$, the problem (N_λ) admits at least one nontrivial weak solution $\bar{u}$ such that $\|\bar{u}\| < \frac{c}{\gamma}$.

Proof. Our aim is to apply Theorem 3.1. To this end, we pick $c_1 = 0$ and $c_2 = c$. From (8) one has

$$a(c_2,d) = \frac{\max\limits_{|t|\leq c}F(t) - F(d)}{\delta_2^2c^2-\gamma^2d^2} \leq \frac{F(c) - \left(\frac{\gamma^2d^2}{\delta_2^2c^2}F(c)\right)}{\delta_2^2c^2-\gamma^2d^2} = \frac{F(c)}{\delta_2^2c^2}.$$

On the other hand, one has $b(c_1,d) = \dfrac{F(d)}{\gamma^2 d^2}$. Hence, owing to (8), Theorem 3.1 ensures the conclusion.

$\square$

Now, we point out the following relevant consequence of Corollary 3.2.

Theorem 3.3. *Let $\alpha : [a,b] \to \mathbb{R}$ be a nonnegative function and $f,h : \mathbb{R} \to \mathbb{R}$ be continuous functions. Assume that (i) holds. Assume that*

$$\lim_{\xi \to 0^+} \frac{f(\xi)}{\xi} = +\infty, \tag{9}$$

and put $\lambda^* = \dfrac{b-a}{2\|\alpha\|_1}\left(\dfrac{\delta_2}{\gamma}\right)^2 \sup\limits_{c>0} \dfrac{c^2}{F(c)}$. Then, for each $\lambda \in]0,\lambda^*[$, the problem (N_λ) admits at least one positive weak solution.

Proof. Fix $\lambda \in]0,\lambda^*[$. Then, there is $c > 0$ such that $\lambda < \dfrac{b-a}{2\|\alpha\|_1}\left(\dfrac{\delta_2}{\gamma}\right)^2 \dfrac{c^2}{F(c)}$. From (9) there is $d < \dfrac{\delta_2}{\gamma}c$ such that $\dfrac{2\|\alpha\|_1}{b-a}\dfrac{F(d)}{d^2} > \dfrac{1}{\lambda}$. Hence, Corollary 3.2 ensures the conclusion. $\square$

Remark 3.4. Taking (9) into account, fix $\rho > 0$ such that $f(\xi) > 0$ for all $\xi \in]0,\rho[$. Then, put $\bar\lambda = \dfrac{b-a}{2\|\alpha\|_1}\left(\dfrac{\delta_2}{\gamma}\right)^2 \sup_{c\in]0,\rho[} \dfrac{c^2}{F(c)}$. Clearly, $\bar\lambda \le \lambda^*$. Now, fixed $\lambda \in \left]0,\bar\lambda\right[$ and arguing as in the proof of Theorem 3.3, there are $c \in]0,\rho[$ and $d < \dfrac{\delta_2}{\gamma}c$ such that $\dfrac{b-a}{2\|\alpha\|_1}\dfrac{d^2}{F(d)} < \lambda < \dfrac{b-a}{2\|\alpha\|_1}\left(\dfrac{\delta_2}{\gamma}\right)^2\dfrac{c^2}{F(c)}$. Hence, Corollary 3.2 ensures that, for each $\lambda \in \left]0,\bar\lambda\right[$, the problem (N_λ) admits at least one positive weak solution $\bar u_\lambda$ such that

$$|\bar u_\lambda(x)| < \frac{\rho}{\gamma},$$

for all $x \in [a,b]$.

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